Backpaper examination
First semester 2009
B.Math.Hons.First year
Algebra I — B.Sury
Answer any 5

Q 1.
For a prime $p > 2$, show that exactly half of the elements in the group $\mathbf{Z}_p^*$ are squares.
Hint : Use the squaring map.

Q 2.
If H is any subgroup of finite index in a group G, use the action of G on the set of left cosets of H to prove that H contains a normal subgroup of finite index in G.

Q 3.
Prove that if a p-group P acting on a set with N elements where $p \not| N$, then P fixes a point.

Q 4.
If P is a p-Sylow subgroup of a finite group G, and if N denotes the normalizer of P in G, prove that the normalizer of N in G is N itself.
Hint : You may use Sylow's second theorem for the group N.

Q 5.
Define the product IJ of two ideals I, J in a commutative ring. Prove that

$$IJ \subset I \cap J \subset I, J \subset I + J.$$

Give examples of ideals I, J in the ring $\mathbf{Z}$ to show that each of these inclusions can be proper.

Q 6.
If A is an integral domain (with unity), prove that $A[X]$ is a PID if and only if A is a field.

Q 7.
Let n be a natural number. Describe explicitly, the smallest subring of $\mathbf{C}$ which contains all the n-th roots of unity.